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On the Connection Between Zwanzig's Classical Projection Operator Formalism and Coarse-Graining

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It is pointed out that the following two procedures are identical: (i) applying Zwanzig's projection operator to the distribution function; (ii) coarse-graining the distribution function with cells which are defined by means of the macroscopic variables and their inaccuracies, and going to the limit of vanishing inaccuracies.

KEY WORDS: Projection operator; coarse-graining.

Some years ago, Zwanzig introduced projection operator techniques into statistical mechanics,[1,2] which have been shown to be very powerful tools for the derivation of reduced equations from first principles (see, e.g., Ref. 3 and references cited therein). The most general explicit form of the classical projection operator P has been given as[2]

$$G_1(x) := PG(x)$$

$$:= \int dx' \prod_{i=1}^{r} \delta(A_i(x') - A_i(x))G(x') \Big/ \int dx' \prod_{i=1}^{r} \delta(A_i(x') - A_i(x)) \qquad (1)$$

where the $A_i(x)$, $i = 1,..., r$, are the phase functions describing the macroscopic variables. By construction, $G_1(x)$ is constant on the hypersurface $A_i(x) = a_i$, $i = 1,..., r$. This allows us to write

$$G_1(x)\Big|_{\substack{A_i(x)=a_i \\ i=1,...,r}} = \int dx \prod_{i=1}^{r} \delta(A_i(x) - a_i)G(x) \Big/ \int dx \prod_{i=1}^{r} \delta(A_i(x) - a_i) \qquad (2)$$

It is trivial to see that

$$\int dx \prod_{i=1}^{r} \delta(A_i(x) - a_i)G(x) = \frac{\partial^r}{\partial a_1 \cdots \partial a_r} \int_{\substack{A_i(x) \leq a_i \\ i=1,...,r}} G(x)\, dx \qquad (3)$$

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from which we get immediately

$$G_1(x)\Big|_{\substack{A_i(x)=a_i \\ i=1,\dots,r}} = \frac{\partial^r}{\partial a_1 \cdots \partial a_r} \int_{\substack{A_i(x)\leqslant a_i \\ i=1,\dots,r}} G(x)\, dx \Big/ \frac{\partial^r}{\partial a_1 \cdots \partial a_r} \int_{\substack{A_i(x)\leqslant a_i \\ i=1,\dots,r}} dx \tag{4}$$

We have recently shown[4] that the familiar coarse-graining

$$G_{cg}(x)|_{x\in\Omega_j} := \int_{\Omega_j} G(x)\, dx \Big/ \int_{\Omega_j} dx \tag{5}$$

with the cells introduced by van Kampen[5]

$$\Omega = \{x | a_i \leqslant A_i(x) \leqslant a_i + \delta_i, \quad i = 1,\dots, r\} \tag{6}$$

leads in the limit of vanishing inaccuracies δ_i to expression (4), i.e.,

$$\lim_{\delta_i \to 0;\, i=1,\dots,r} G_{cg}(x) = G_1(x) \tag{7}$$

This shows that Zwanzig's classical projection is identical with coarse-graining in the limit of vanishing inaccuracies of the macroscopic variables.

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